THE KANSAS UNIVERSITY SCIENCE BULLETIN.

VOL. XIII, NO. 5—MAY, 1920.

CONTENTS:

A CALCULATION OF THE INVARIANTS AND COVARIANTS FOR RULED SURFACES,
E. B. Stouffer.

PUBLISHED BY THE UNIVERSITY,

LAWRENCE, KAN.

Entered at the post-office in Lawrence as second-class matter.

9–860

THE KANSAS UNIVERSITY SCIENCE BULLETIN.

VOL. XIII.] MAY, 1920. [No. 5.

A Calculation of the Invariants and Covariants for Ruled Surfaces.*

BY E. B. STOUFFER.

IN Wilczynski's *Projective Differential Geometry of Curves and Ruled Surfaces* † it is shown that the projective differential properties of a non-developable ruled surface may be studied by means of a system of differential equations of the form ‡

$$(A)\begin{cases} y''+2p_{11}y'+2p_{12}z'+q_{11}y+q_{12}z=0, \\ z''+2p_{21}y'+2p_{22}z'+q_{21}y+q_{22}z=0, \end{cases}$$

where p_{ik} and q_{ik} are functions of the independent variable x. The most general transformations leaving (A) unchanged in form are given by the equations

$$(1)\begin{cases} y=a_{11}\bar{y}+a_{12}\bar{z}, \\ z=a_{21}\bar{y}+a_{22}\bar{z}, \end{cases} \qquad \triangle \equiv a_{11}a_{22}-a_{12}a_{21}\neq 0,$$

$$(2) \qquad \xi=\xi(x),$$

where a_{ik} and ξ are arbitrary functions of x.

A function of the coefficients of (A) and their derivatives and of the dependent variables and their derivatives which remains unchanged in value by the transformation (1) is called a *semi-covariant* and if it remains unchanged in value also by the transformation (2) it is called a *covariant*. Semi-covariants or covariants which do not involve the dependent variables or their derivatives are called *seminvariants* or *invariants*, respectively. The invariants and covariants of system (A) are used in the study of the

* Received for publication May 10.

† Teubner, Leipzig, 1906.

‡ Wilczynski writes his system without the factor 2 in the coefficients of y' and z'. Its introduction makes some of the results appear in simpler form.

properties of ruled surfaces. Their calculation as given by Wilczynski involves the solution of several rather complicated systems of partial differential equations. It is the purpose of this paper to obtain the same results by much shorter methods.

In 1915 Green published a paper* in which he obtains the invariants and covariants of the general form of the system of partial differential equations associated with curved surfaces from the invariants and covariants of a canonical form of these equations. Green points out that his general method is of wide application. This scheme of making the calculations first for a simplified system and then transforming to the coefficients of the original system is used in the present paper.

The results in this paper carry the same label as do the corresponding results in Wilczynski's book but there are differences in numerical coefficients and in signs because of the introduction of the binomial coefficients in equations (A) and because of a change of sign in the defining expression for u_{ik}.

1. The Semi-Canonical Form.

Let us make the transformation (1) upon the system (A). There immediately results the system

$$(3)\quad \begin{cases} a_{11}\bar{y}'' + a_{12}\bar{z}'' + 2\,(a'_{11} + p_{11}\,a_{11} + p_{12}\,a_{21})\bar{y}' + \\ \qquad\qquad 2\,(a'_{12} + p_{11}\,a_{12} + p_{12}\,a_{22})\,\bar{z}' \\ \quad + (a''_{11} + 2\,p_{11}\,a'_{11} + 2\,p_{12}\,a'_{21} + q_{11}a_{11} + q_{12}\,a_{21})\,\bar{y} \\ \quad + (a''_{12} + 2\,p_{11}\,a'_{12} + 2\,p_{12}\,a'_{22} + q_{11}\,a_{12} + q_{12}\,a_{22})\,\bar{z} = 0, \\ a_{21}\bar{y}'' + a_{22}\bar{z}'' + 2\,(a'_{21} + p_{21}\,a_{11} + p_{22}\,a_{21})\,\bar{y}' + \\ \qquad\qquad 2\,(a'_{22} + p_{21}\,a_{12} + p_{22}\,a_{22})\,\bar{z}' \\ \quad + (a''_{21} + 2\,p_{21}\,a'_{11} + 2\,p_{22}\,a'_{21} + q_{21}\,a_{11} + q_{22}\,a_{21})\,\bar{y} \\ \quad + (a''_{22} + 2\,p_{21}\,a'_{12} + 2\,p_{22}\,a'_{22} + q_{21}\,a_{12} + q_{22}\,a_{22})\,\bar{z} = 0. \end{cases}$$

If a_{ik} are so chosen that

$$(4)\quad a'_{ik} = -\sum_{j=1}^{2} p_{ij}\,a_{jk},\ (i, k = 1, 2),$$

the coefficients of $\bar{y}'$ and $\bar{z}'$ in (3) vanish. Such a solution for a_{ik} is always possible since it is equivalent merely to choosing (a_{11}, a_{21}) and (a_{12}, a_{22}) as two distinct pairs of solutions of the system of differential equations

$$\rho' = -\,(p_{11}\,\rho + p_{12}\,\sigma)$$
$$\sigma' = -\,(p_{21}\,\rho + p_{22}\,\sigma).$$

*G. M. Green, *On the Theory of Curved Surfaces, and Canonical Systems in Projective Differential Geometry.* Transactions of the American Mathematical Society, Vol. 16 (1915), pp. 1-12.

The substitution from (4) into (3) now gives

$$(5)\quad \begin{cases} a_{11}\bar{y}'' + a_{12}\bar{z}'' + (u_{11}a_{11} + u_{12}a_{21})\bar{y} + (u_{11}a_{12} + u_{12}a_{22})\bar{z} = 0, \\ a_{21}\bar{y}'' + a_{22}\bar{z}'' + (u_{21}a_{11} + u_{22}a_{21})\bar{y} + (u_{21}a_{12} + u_{22}a_{22})\bar{z} = 0, \end{cases}$$

where*

$$(6)\quad u_{ik} = q_{ik} - p'_{ik} - \sum_{j=1}^{2} p_{ij}p_{jk}, \quad (i, k = 1, 2).$$

The system (5) may be put into the form

$$(B)\begin{cases} \bar{y}'' + \bar{q}_{11}\bar{y} + \bar{q}_{12}\bar{z} = 0, \\ \bar{z}'' + \bar{q}_{21}\bar{y} + \bar{q}_{22}\bar{z} = 0, \end{cases}$$

if we write

$$(7)\quad \triangle\bar{q}_{ik} = \sum_{l=1}^{2}\sum_{j=1}^{2} A_{ji}a_{lk}u_{jl}, \quad (i, k = 1, 2),$$

where A_{ji} is the algebraic minor a_{ji} in the determinant of the transformation (1). Wilczynski calls (B) the semi-canonical form of the system (A).

The differentiation of equations (7) gives

$$(8)\quad \triangle\bar{q}'_{ik} = \sum_{l=1}^{2}\sum_{j=1}^{2}[A_{ji}a_{lk}u'_{jl} + A_{ji}a'_{lk}u_{jl} + A'_{ji}a_{lk}u_{jl}] - \triangle'\bar{q}_{ik}, \ (i, k = 1, 2).$$

By the use of (4) we find

$$\sum_{j=1}^{2} A'_{ji}u_{jl} = \sum_{j=1}^{2} A_{ji}\left[-(p_{11} + p_{22})u_{jl} + \sum_{m=1}^{2} p_{jm}u_{ml}\right],$$

$$\triangle' = -(p_{11} + p_{22})\triangle,$$

whence it follows at once that

$$(9)\quad \triangle\bar{q}'_{ik} = \sum_{l=1}^{2}\sum_{j=1}^{2} A_{ji}a_{lk}v_{jl}, \qquad (i, k = 1, 2),$$

where

$$(10)\quad v_{ik} = u'_{ik} + \sum_{j=1}^{2}(p_{ij}u_{jk} - p_{jk}u_{ij}), \quad (i, k = 1, 2).$$

It follows without calculation that

$$(11)\quad \triangle\bar{q}''_{ik} = \sum_{l=1}^{2}\sum_{j=1}^{2} A_{ji}a_{lk}w_{jl}, \quad (i, k = 1, 2),$$

where

$$(12)\quad w_{ik} = v'_{ik} + \sum_{j=1}^{2}(p_{ij}v_{jk} - p_{jk}v_{ij}), \quad (i, k = 1, 2).$$

---

*The expression here used for u_{ik} differs in sign as well as in numerical coefficients from that used by Wilczynski.

Let us rewrite transformations (1) and (2) in the form

$$(13)\quad \begin{cases} \bar{y} = \beta_{11} Y + \beta_{12} Z, \\ \bar{z} = \beta_{21} Y + \beta_{22} Z, \end{cases} \qquad \beta_{11}\beta_{22} - \beta_{12}\beta_{21} \neq 0.$$

$$(14)\quad \xi = \xi(x),$$

and find the most general nature which these transformations may have and still leave (B) in the semi-canonical form. By these transformations (B) is converted into

$$(15)\quad \begin{cases} \beta_{11}(\xi')^2 \dfrac{d^2 Y}{d\xi^2} + \beta_{12}(\xi')^2 \dfrac{d^2 Z}{d\xi^2} + (\beta_{11}\xi'' + 2\beta'_{11}\xi')\dfrac{dY}{d\xi} + \\ \qquad (\beta_{12}\xi'' + 2\beta'_{12}\xi')\dfrac{dZ}{d\xi} + (\beta''_{11} + \bar{q}_{11}\beta_{11} + \bar{q}_{12}\beta_{21})Y + \\ \qquad\qquad (\beta''_{12} + \bar{q}_{11}\beta_{12} + \bar{q}_{12}\beta_{22})Z = 0, \\ \beta_{21}(\xi')^2 \dfrac{d^2 Y}{d\xi^2} + \beta_{22}(\xi')^2 \dfrac{d^2 Z}{d\xi^2} + (\beta_{21}\xi'' + 2\beta'_{21}\xi')\dfrac{dY}{d\xi} + \\ \qquad (\beta_{22}\xi'' + 2\beta'_{22}\xi')\dfrac{dZ}{d\xi} + (\beta''_{21} + \bar{q}_{21}\beta_{11} + \bar{q}_{22}\beta_{21})Y + \\ \qquad\qquad (\beta''_{22} + \bar{q}_{21}\beta_{12} + \bar{q}_{22}\beta_{22})Z = 0. \end{cases}$$

This system is in the form of system (B) if and only if

$$\beta_{ij}\xi'' + 2\beta_{ij}\xi' = 0, \quad (i,j = 1,2),$$

that is, if

$$(16)\quad \beta_{ij} = \frac{b_{ij}}{\sqrt{\xi'}}, \quad (i,j = 1,2),$$

where b_{ij} are constants. If these values for β_{ij} are substituted into (15) that system may be written in the form

$$(C)\quad \begin{cases} \dfrac{d^2 Y}{d\xi^2} + Q_{11}Y + Q_{12}Z = 0, \\ \dfrac{d^2 Z}{d\xi^2} + Q_{21}Y + Q_{22}Z = 0, \end{cases}$$

if we put

$$(17)\quad DQ_{ik} + \frac{1}{(\xi')^2}\sum_{j=1}^{2} B_{ji}\left[\left(\tfrac{1}{4}\eta^2 - \tfrac{1}{2}\eta'\right)b_{ik} + \sum_{l=1}^{2} b_{lk}\bar{q}_{jl}\right], \quad (i,k = 1,2),$$

where $\eta = \dfrac{\xi''}{\xi'}$ and where B_{ji} is the minor of b_{ji} in the determinant $D \equiv b_{11}b_{22} - b_{12}b_{21}$.

The transformations

$$(18)\begin{cases} \bar{y} = \dfrac{b_{11}}{\sqrt{\xi'}} Y + \dfrac{b_{12}}{\sqrt{\xi'}} Z, \\ \bar{z} = \dfrac{b_{21}}{\sqrt{\xi'}} Y + \dfrac{b_{22}}{\sqrt{\xi'}} Z, \\ \xi = \xi(x). \end{cases}$$

which leave B unchanged in form may be considered as consisting of the transformation

$$(19)\begin{cases} \bar{y} = b_{11} Y + b_{12} Z, \\ z = b_{21} Y + b_{22} Z, \end{cases} \qquad D \equiv b_{11} b_{22} - b_{12} b_{21} \neq 0,$$

in which $\xi = x$, and of the transformations

$$(20)\begin{cases} \bar{y} = \dfrac{1}{\sqrt{\xi'}} Y, \\ \bar{z} = \dfrac{1}{\sqrt{\xi'}} Z, \\ \xi = \xi(x), \end{cases}$$

in which $b_{11} = b_{22} = 1$ and $b_{12} = b_{21} = 0$.

2. The Seminvariants.

Let us first find those functions of the coefficients of (B) and their derivatives which remain unchanged in value by the transformation (19). Equations (17) show that (19) converts $\bar{q}_{ik}$ into Q_{ik} where

$$(21) \quad D Q_{ik} = \sum_{l=1}^{2} \sum_{j=1}^{2} B_{ji} b_{lk} \bar{q}_{jl}, \quad (i, k = 1, 2).$$

If the transformation (19) is made infinitesimal by putting $b_{ii} = 1 + \varphi_{ii}\, \delta t$ and $b_{ij} = \varphi_{ij}\, \delta t, (i \neq j)$, where φ_{ij} are arbitrary constants and δt an infinitesimal, the infinitesimal transformations of $\bar{q}_{ik}$ are found from (21) to be

$$(22) \quad \delta \bar{q}_{ik} = \sum_{j=1}^{2} (\varphi_{jk} \bar{q}_{ij} - \varphi_{ij} \bar{q}_{jk}) \delta t, \quad (i, k = 1, 2).$$

In accordance with the Lie theory the desired functions must satisfy the system of partial differential equations.

$$(23) \quad U_{rs} f \equiv \sum_{l=1}^{2} (\bar{q}_{lr} \frac{\delta f}{\delta \bar{q}_{ls}} - \bar{q}_{sl} \frac{\delta f}{\delta \bar{q}_{rl}}) = 0, \quad (r, s = 1, 2).$$

Between these four equations there are the two relations

$$(24) \quad U_{11} + U_{22} = 0,$$

$$(25) \qquad \bar{q}_{12} U_{12} + \bar{q}_{21} U_{21} + \bar{q}_{11} U_{11} + \bar{q}_{22} U_{22} = 0.$$

Since the system contains four variables there are just two solutions. These are easily seen to be

$$I = \bar{q}_{11} + \bar{q}_{22}, \quad J = \bar{q}_{11}\bar{q}_{22} - \bar{q}_{12}\bar{q}_{21}.$$

Since the coefficients in (19) are constants the transformations of the various derivatives of $\bar{q}_{ik}$ will be of exactly the same form as the transformations of $\bar{q}_{ik}$. The differential equations for the functions involving $\bar{q}'_{ik}$ as well as $\bar{q}_{ik}$ are simply (23) with terms of the same form in $\bar{q}'_{ik}$ added. The relations (25) ceases to hold so that there are just three more solutions. These are evidently

$$I', J', K = \bar{q}'_{11}\bar{q}'_{22} - \bar{q}'_{12}\bar{q}'_{21}.$$

In the system of equations for the functions involving also $\bar{q}''_{ik}$ there are just three independent equations and four more variables so that there are four more solutions. These are evidently

$$I'', J'', K', L = \bar{q}''_{11}\bar{q}''_{22} - \bar{q}''_{12}\bar{q}''_{21}.$$

A continuation of this process shows that all the desired functions involving higher derivatives of $\bar{q}_{ik}$ can be obtained by forming the successive derivatives of I, J, K, L.

Let us now substitute in I, J, K, L and their derivatives the expressions for $\bar{q}_{ik}, \bar{q}'_{ik}, \bar{q}''_{ik}$ given in (7), (9) and (11). A comparison of these equations with (21) and its derivatives shows that $\bar{q}_{ik}$ is expressed in terms of u_{ik}, $\bar{q}'_{ik}$ in terms of v_{ik}, and $\bar{q}''_{ik}$ in terms of w_{ik} in exactly the same way that Q_{ik} is expressed in terms of $\bar{q}_{ik}$, Q'_{ik} in terms $\bar{q}'_{ik}$, and Q''_{ik} in terms of $\bar{q}''_{ik}$, respectively, except of course that a_{lk} replaces b_{lk}. If now in I, J, K, L or in their derivatives we replace $\bar{q}_{ik}, \bar{q}'_{ik}, \bar{q}''_{ik}$ by $Q_{ik}, Q'_{ik}, Q''_{ik}$ respectively, we obtain the original functions of $\bar{q}_{ik}, \bar{q}'_{ik}, \bar{q}''_{ik}$. It follows therefore that if in I, J, K, L and their derivatives we replace $\bar{q}_{ik}, \bar{q}'_{ik}, \bar{q}''_{ik}$ by u_{ik}, v_{ik}, w_{ik}, respectively, we obtain the result of substituting (7), (9), (11) into these functions. In other words

$$(26) \begin{cases} I = u_{11} + u_{22}, & J = u_{11}u_{22} - u_{12}u_{21}, \\ I' = v_{11} + v_{22}, & J' = u_{11}v_{22} + u_{22}v_{11} - u_{12}v_{21} - u_{21}v_{12}, \\ I'' = w_{11} + w_{22}, & J'' = 2K + u_{11}w_{22} + u_{22}w_{11} - u_{12}w_{21} - u_{21}w_{12}, \\ K = v_{11}v_{22} - v_{12}v_{21}, & L = w_{11}w_{22} - w_{12}w_{21}, \\ K' = v_{11}w_{22} + v_{22}w_{11} - v_{12}w_{21} - v_{21}w_{12}. \end{cases}$$

The expressions (26) and their derivatives are all seminvariants of the system (A) and moreover they form a complete

system of seminvariants for the system (A). To show these facts let us suppose that we have two systems of form (A) which are equivalent under a transformation of form (1). Each of these systems may be reduced to a semi-canonical form and these must be equivalent under a transformation of form (19). A seminvariant expression, $\bar{q}_{11} + \bar{q}_{22}$, say, formed for these two semi-canonical forms must be equal and each is equal to the expression $u_{11} + u_{22} = I$ formed for its corresponding original system. Therefore the two expressions for I are equal and I must be a seminvariant. The same reasoning applies to the other expressions (26). That we have a complete system of seminvariants is obvious from the fact that every seminvariant of (A) must have a semi-canonical form which remains unchanged by transformations which leave the semi-canonical form invariant.

3. The Semi-Covariants.

We shall now find the semi-covariants of (A) by finding first the semi-canonical form of these semi-covariants. The transformation (1) when solved for $\bar{y}$ and $\bar{z}$ has the form

$$(27)\quad \begin{cases} \Delta \bar{y} = a_{22}\, y - a_{12}\, z, \\ \Delta \bar{z} = -\, a_{21}\, y + a_{11}\, z. \end{cases}$$

When the coefficients of this transformation are subjected to the conditions (4) we find

$$(28)\quad \begin{cases} \Delta \bar{y}' = a_{22}\, \rho - a_{12}\, \sigma, \\ \Delta \bar{z}' = -\, a_{21}\, \rho + a_{11}\, \sigma. \end{cases}$$

where

$$(29)\quad \rho = y' + p_{11}\, y + p_{12}\, z, \quad \sigma = z' + p_{21}\, y + p_{22}\, z.$$

Evidently semi-covariants need contain no higher derivatives of y and z than the first.

The semi-canonical form of the semi-covariants will be found by subjecting (B) to the transformation (19). Since the coefficients in (19) are constants

$$(30)\quad \begin{cases} \bar{y}' = b_{11}\, Y' + b_{12}\, Z', \\ \bar{z}' = b_{21}\, Y' + b_{22}\, Z', \end{cases}$$

and it follows at once that

$$(31)\quad P = \bar{y}\, \bar{z}' - \bar{y}'\, \bar{z}$$

is a semi-covariant.

The system of differential equations for the semi-canonical form of the semi-covariants is the same as the system for the semi-canonical form of the seminvariants except that each equation contains more terms and there are four more variables. The relations (24) and (25) both cease to hold so that there are three semi-covariants or four relative semi-covariants.

Equations (19) and (21) show that the expressions $\bar{q}_{11}\bar{y} + \bar{q}_{12}\bar{z}$ and $\bar{q}_{21}\bar{y} + \bar{q}_{22}\bar{z}$ are transformed cogrediently with $\bar{y}$ and $\bar{z}$, respectively. The same is of course true of $\bar{q}'_{11}\bar{y} + \bar{q}'_{12}\bar{z}$ and $\bar{q}'_{21}\bar{y} + \bar{q}'_{22}\bar{z}$, respectively. It follows at once that the three expressions

$$(32)\begin{cases} C = (\bar{q}_{11}\bar{y} + \bar{q}_{12}\bar{z})\,\bar{z} - (\bar{q}_{21}\bar{y} + \bar{q}_{22}\bar{z})\,\bar{y}, \\ E = (\bar{q}'_{11}\bar{y} + \bar{q}'_{12}\bar{z})\,\bar{z} - (\bar{q}'_{21}\bar{y} + \bar{q}'_{22}\bar{z})\,\bar{y}, \\ O = (\bar{q}_{11}\bar{y} + \bar{q}_{12}\bar{z})\,\bar{z}' - (\bar{q}_{21}\bar{y} + \bar{q}_{22}\bar{z})\,\bar{y}', \end{cases}$$

are independent relative semi-covariants. A comparison of (19) and (30) with (27) and (28) shows that the semi-covariants (31) and (32) can be expressed in terms of the original variables and coefficients if $\bar{y}$ is replaced by y, $\bar{z}$ by z, $\bar{y}'$ by ρ and $\bar{z}'$ by σ at the same time that $\bar{q}_{ik}$ and $\bar{q}'_{ik}$ are replaced by u_{ik} and v_{ik}, respectively. Thus we have

$$(33)\begin{cases} P = y\sigma - z\rho, \\ C = (u_{11}y + u_{12}z)\,z - (u_{21}y + u_{22}z)\,y, \\ E = (v_{11}y + v_{12}z)\,z - (v_{21}y + v_{22}z)\,y, \\ O = (u_{11}y + u_{12}z)\sigma - (u_{21}y + u_{22}z)\,\rho. \end{cases}$$

By the same argument as in the case of seminvariants these four semi-covariants are known to form a complete system for (A).

4. The Canonical Form and the Invariants.

We shall now proceed to find those functions of the seminvariants in their semi-canonical form which remain unchanged except for a factor $\frac{1}{(\xi')^m}$ by the transformation (20). We shall thus obtain the functions of the coefficients of (B) and their derivatives which remain unchanged by (18), except for the factor $\frac{1}{(\xi')^m}$.

Equation (17) shows that (20) converts (B) into a new system whose coefficients Q_{ik} are given by the equations

$$(34)\begin{cases} Q_{ii} = \frac{1}{(\xi')^2}\left(\frac{1}{4}\eta^2 - \frac{1}{2}\eta' + \bar{q}_{ii}\right), & (i = 1, 2), \\ Q_{ik} = \frac{1}{(\xi')^2}\,\bar{q}_{ik}, & (i, k = 1, 2;\ i \neq k). \end{cases}$$

We notice that

$$Q_{11} + Q_{22} = \frac{1}{(\xi')^2}\left(\frac{1}{2}\eta^2 - \eta' + \bar{q}_{11} + \bar{q}_{12}\right),$$

so that $Q_{11} + Q_{22} = 0$, provided that

$$(35) \qquad \mu \equiv \eta' - \tfrac{1}{2}\eta^2 = \bar{q}_{11} + \bar{q}_{22}.$$

From equations (34) we have at once, if (35) is satisfied,

$$(36)\begin{cases} Q_{ii} = \frac{1}{(\xi')^2}\left(\bar{q}_{ii} - \frac{1}{2}I\right), & (i = 1, 2), \\ Q_{ik} = \frac{1}{(\xi')^2}\,\bar{q}_{ik}, & (i, k = 1, 2;\ i \neq k), \end{cases}$$

whence

$$(37)\begin{cases} Q'_{ii} = \frac{1}{(\xi')^3}\left[\bar{q}'_{ii} - \frac{1}{2}I' - 2\eta\left(\bar{q}_{ii} - \frac{1}{2}I\right)\right], \\ Q''_{ii} = \frac{1}{(\xi')^4}\left[\bar{q}''_{ii} - \frac{1}{2}I'' + I^2 - 2I\bar{q}_{ii} - 5\eta\left(\bar{q}'_{ii} - \frac{1}{2}I'\right) + \right. \\ \qquad\qquad \left. 5\eta^2\left(\bar{q}_{ii} - \frac{1}{2}I\right)\right], \\ Q'_{ik} = \frac{1}{(\xi')^3}\left(\bar{q}'_{ik} - 2\eta\,\bar{q}_{ik}\right), \quad (i, k = 1, 2;\ i \neq k), \\ Q''_{ik} = \frac{1}{(\xi')^4}\left(\bar{q}''_{ik} - 2I\bar{q}_{ik} - 5\eta\,\bar{q}'_{ik} + 5\eta^2 q_{ik}\right). \end{cases}$$

Let us now assume that (B) has been converted into

$$(D)\begin{cases} \bar{y}'' + Q_{11}\bar{y} + Q_{12}\bar{z} = 0, \\ \bar{z}'' + Q_{21}\bar{y} + Q_{22}\bar{z} = 0, \end{cases}$$

where Q_{ik} have the values (36) so that $Q_{11} + Q_{22} = 0$. The system (D) is called the canonical form of (A).

If the seminvariants for (D) corresponding to I, J, K, L for (B) are denoted by I_1, J_1, K_1, L_1, respectively, equations (37) show that

$$
(38)\quad \left\{
\begin{aligned}
& I_1 = 0,\ J_1 = \frac{1}{(\xi')^4}\left[J - \tfrac{1}{4}I^2\right], \\
& J'_1 = \frac{1}{(\xi')^5}\left[\frac{d}{dx}(J - \tfrac{1}{4}I^2) - 4\eta(J - \tfrac{1}{4}I^2)\right], \\
& J''_1 = \frac{1}{(\xi')^6}\left[\frac{d^2}{dx^2}(J - \tfrac{1}{4}I^2) - 4I(J - \tfrac{1}{4}I^2) - \right. \\
& \qquad\qquad \left. 9\eta\frac{d}{dx}(J - \tfrac{1}{4}I^2) + 18\eta^2(J - \tfrac{1}{4}I^2)\right], \\
& K_1 = \frac{1}{(\xi')^6}\left[K - \tfrac{1}{4}(I')^2 - 2\eta\frac{d}{dx}(J - \tfrac{1}{4}I^2) + \right. \\
& \qquad\qquad \left. 4\eta^2(J - \tfrac{1}{4}I^2)\right], \\
& K'_1 = \frac{1}{(\xi')^7}\left[\frac{d}{dx}\left\{K - \tfrac{1}{4}(I')^2\right\} - 2I\frac{d}{dx}(J - \tfrac{1}{4}I^2) - \right. \\
& \qquad 6\eta\left\{K - \tfrac{1}{4}(I')^2\right\} - 2\eta\left\{\frac{d^2}{dx^2}(J - \tfrac{1}{4}I^2) - \right. \\
& \qquad \left. 4I(J - \tfrac{1}{4}I^2)\right\} + 15\eta^2\frac{d}{dx}(J - \tfrac{1}{4}I^2) - \\
& \qquad \left. 20\eta^3(J - \tfrac{1}{4}I^2)\right], \\
& I_1 = \frac{1}{(\xi')^8}\left[L - \tfrac{1}{4}(I'')^2 + 4I\left\{K - \tfrac{1}{4}(I')^2\right\}\right. \\
& \qquad - 2I\frac{d^2}{dx^2}(J - \tfrac{1}{4}I^2) + 4I^2(J - \tfrac{1}{4}I^2) \\
& \qquad - 5\eta\left\{\frac{d}{dx}[K - \tfrac{1}{4}(I')^2] - 2I\frac{d}{dx}(J - \tfrac{1}{4}I^2)\right\} \\
& \qquad + 5\eta^2\left\{\frac{d^2}{dx^2}(J - \tfrac{1}{4}I^2) - 4I(J - \tfrac{1}{4}I^2)\right\} \\
& \qquad + 15\eta^2\left\{K - \tfrac{1}{4}(I')^2\right\} - 25\eta^3\frac{d}{dx}(J - \tfrac{1}{4}I^2) \\
& \qquad \left. + 25\eta^4(J - \tfrac{1}{4}I^2)\right].
\end{aligned}
\right.
$$

The system (D) is left in the canonical form by the transformation (20) provided that $\mu = 0$. We shall now seek those functions of the seminvariants in their semi-canonical form which are left unchanged in value by the transformation (20) subject to the condition $\mu = 0$.

From (34) or by direct substitution we find that (20) with $\mu = 0$ converts Q_{ik} into

$$
(39)\qquad \overline{Q}_{ik} = \frac{1}{(\xi')^2}Q_{ik}, \quad (i, k = 1, 2),
$$

whence it follows that

$$(40)\begin{cases} \bar{Q}'_{ik} = \frac{1}{(\xi')^3}(Q'_{ik} - 2\eta Q_{ik}), \\ \bar{Q}''_{ik} = \frac{1}{(\xi')^4}(Q''_{ik} - 5\eta Q'_{ik} + 5\eta^2 Q_{ik}). \end{cases} \quad (i, k = 1, 2),$$

These results show by direct substitution and by differentiation that J_1, K_1, L_1, and their derivatives for the transformed equations have the values

$$(41)\begin{cases} \bar{J}_1 = \frac{1}{(\xi')^4} J_1, \bar{J}'_1 = \frac{1}{(\xi')^5}(J'_1 - 4\eta J_1), \\ \bar{J}''_1 = \frac{1}{(\xi')^6}(J''_1 - 9\eta J'_1 + 18\eta J_1), \\ \bar{K}_1 = \frac{1}{(\xi')^6}(K_1 - 2\eta J'_1 + 4\eta^2 J_1), \\ \bar{K}'_1 = \frac{1}{(\xi')^7}(K'_1 - 6\eta K_1 - 2\eta J''_1 + 15\eta^2 J'_1 - 20\eta^3 J_1), \\ \bar{L}_1 = \frac{1}{(\xi')^8}(L_1 - 5\eta K'_1 + 5\eta^2 J''_1 + 15\eta^2 K_1 - 25\eta^3 J'_1 \\ \qquad\qquad + 25\eta^4 J_1). \end{cases}$$

If the transformation (20) is made infinitesimal by putting

$$\xi = x + \varphi(x)\,\delta t$$

where $\varphi(x)$ is an arbitrary function of x and δt is an infinitesimal, the infinitesimal transformations of J_1, K_1, L_1, and their derivatives are found by direct substitution in (41) to be

$$(42)\begin{cases} \delta J_1 = -4\varphi' J_1 \delta t, \\ \delta J'_1 = (-5\varphi' J'_1 - 4\varphi'' J_1)\,\delta t, \\ \delta J''_1 = (-6\varphi' J''_1 - 9\varphi'' J'_1)\,\delta t, \\ \delta K_1 = (-6\varphi' K_1 - 2\varphi'' J'_1)\,\delta t, \\ \delta K'_1 = (-7\varphi' K'_1 - 6\varphi'' K_1 - 2\varphi'' J''_1)\,\delta t, \\ \delta L_1 = (-8\varphi' L_1 - 5\varphi'' K'_1)\,\delta t. \end{cases}$$

The resulting system of partial differential equations whose solutions are invariants of (D) under the transformation (20) with $\mu = 0$ contains two independent equations. There are therefore four such absolute invariants involving the variables J_1, J'_1, J''_1, K_1, K'_1, L_1. The five relative invariants may be taken to be

$$(43)\begin{cases} \bar{\theta}_4 = J_1, \bar{\theta}_{4.1} = 9(J'_1)^2 - 8J_1 J''_1, \\ \bar{\theta}_{10} = (J'_1)^2 - 4J_1 K_1, \bar{\theta}_{15} = 5\bar{\theta}_{10} J'_1 - 2\bar{\theta}'_{10} J_1, \\ \bar{\theta}_{18} = \{(J'_1)^2 - 4J_1 K_1\} L + K_1 (J''_1 - 2K_1)^2 + J_1 (K'_1)^2 \\ \qquad\qquad - J'_1 K'_1 (J''_1 - 2K_1). \end{cases}$$

The system of equations for the invariants involving also the next higher derivatives of J_1, K_1, L_1, contains no more equations but three more variables. The three solutions may be taken to be

$$(44)\begin{cases} 4\,J_1\,\overline{\theta}'_{4.1} - 9\,J'_1\,\overline{\theta}_{4.1}, \\ 4\,J_1\,\overline{\theta}'_{15} - 15\,J'_1\,\overline{\theta}_{15}, \\ 4\,J_1\,\overline{\theta}_{18} - 18\,J'_1\,\overline{\theta}_{18}. \end{cases}$$

The invariants involving the next higher derivatives of J_1, K_1, L_1, may obviously be obtained by combining J_1 and J'_1 with the invariants (44). A continuation of this process evidently gives all the independent relative invariants.

The invariants (43) may be expressed in terms of I, J, K, L, and their derivatives by means of (38). However, a comparison of (38) and (41) shows that this substitution can be made, except for a factor $\frac{1}{(\xi')^m}$, by replacing in (43) J_1 by $J - \frac{1}{4}I^2$, J'_1 by $\frac{d}{dx}(J - \frac{1}{4}I^2)$, J''_1 by $\frac{d^2}{dx^2}(J - \frac{1}{4}I^2) - 4\,I\,(J - \frac{1}{4}I^2)$, K_1 by $K - \frac{1}{4}(I')^2$, K'_1 by $\frac{d}{dx}\{K - \frac{1}{4}(I')^2\} - 2\,I\,\frac{d}{dx}(J - \frac{1}{4}I^2)$ and L_1 by $L - \frac{1}{4}(I'')^2 + 4\,I\,\{K - \frac{1}{4}(I')^2\} - 2\,I\,\frac{d^2}{dx^2}(J - \frac{1}{4}I^2) + 4\,I^2(J - \frac{1}{4}I^2)$. The results of these substitutions are as follows:

$$(45)\begin{cases} \theta_4 = J - \frac{1}{4}I^2, \\ \theta_{4.1} = 9\,(\theta'_4)^2 - 8\,\theta_4\,\theta''_4 + 32\,I\,\theta_4^2, \\ \theta_{10} = (\theta'_4)^2 - 4\,\theta_4\,\{K - \frac{1}{4}(I')^2\}, \\ \theta_{15} = 5\,\theta_{10}\,\theta'_4 - 2\,\theta'_{10}\,\theta_4, \\ \theta_{18} = \theta_{10}\,[L - \frac{1}{4}(I'')^2 + 4\,I\,\{K - \frac{1}{4}(I')^2\} - 2\,I\,\theta''_4 + \\ \qquad 4\,I^2\,\theta_4] + \{K - \frac{1}{4}(I')^2\}\,\{\theta''_4 - 4\,I\,\theta_4 - 2\,K + \frac{1}{2}(I')^2\}^2 \\ \qquad + \theta_4\,(K' - \frac{1}{2}I'\,I'' - 2\,I\,\theta'_4)^2 - \\ \qquad \theta'_4\,(K' - \frac{1}{2}I\,I'' - 2\,I\,\theta'_4)\,\{\theta''_4 - 4\,I\,\theta_4 - 2\,K + \frac{1}{2}(I')^2\} \\ \quad = \theta_{10}\,\{L - \frac{1}{4}(I'')^2\} + \{K - \frac{1}{4}(I')^2\}\,(J'' - \frac{1}{2}I\,I'' - 2\,K)^2 \\ \qquad + \theta_4\,(K' - \frac{1}{2}I\,I'')^2 - \theta'_4\,(K' - \frac{1}{2}I\,I'')\,(J'' - \frac{1}{2}I\,I'' - 2\,K). \end{cases}$$

The same reasoning as in the case of the seminvariants shows that the expressions (45) are invariants of (A) and that all independent invariants of (A) are obtained in this way.

There is another expression for an invariant which is easily obtained and which is of geometrical interest. From equation (21) we easily deduce the equations

$$D(Q_{11}-Q_{22}) = (b_{11}b_{22}+b_{12}b_{21})(\bar{q}_{11}-\bar{q}_{22})+2b_{21}b_{22}\bar{q}_{12}-2b_{12}b_{11}\bar{q}_{21},$$
$$DQ_{12} = b_{12}b_{22}(\bar{q}_{11}-\bar{q}_{22})+b_{22}^2\bar{q}_{12}-b_{12}^2\bar{q}_{21},$$
$$DQ_{21} = -b_{21}b_{11}(\bar{q}_{11}-\bar{q}_{22})-b_{21}^2\bar{q}_{12}+b_{11}^2\bar{q}_{21},$$

and exactly similar equations involving derivatives of any order. Thus we know at once that the determinant

$$(46)\qquad \begin{vmatrix} \bar{q}_{11}-\bar{q}_{22} & \bar{q}_{12} & \bar{q}_{21} \\ \bar{q}'_{11}-\bar{q}'_{22} & \bar{q}'_{12} & \bar{q}'_{21} \\ \bar{q}''_{11}-\bar{q}''_{22} & \bar{q}''_{12} & \bar{q}''_{21} \end{vmatrix}$$

is the semi-canonical form of a seminvariant. Furthermore equations (39) and (40) show that it is the semi-canonical form of an invariant. The expression in terms of the original coefficients for this invariant is

$$(47)\qquad \theta_9 = \begin{vmatrix} u_{11}-u_{22} & u_{12} & u_{21} \\ v_{11}-v_{22} & v_{12} & v_{21} \\ w_{11}-w_{22} & w_{12} & w_{21} \end{vmatrix}$$

5. The Covariants.

Let us now return to the semi-canonical form of the semi-covariants and assume that they have been written down for equations (D). If they are denoted by P_1, C_1, B_1, O_1, equations (39) and (40) show that their values for the equations obtained by transforming (D) by (20) with $\mu = 0$ are as follows:

$$\bar{P}_1 = P_1,\quad \bar{C}_1 = \frac{1}{\xi'}C_1,$$
$$\bar{E}_1 = \frac{1}{(\xi')}^2(E_1-2\eta C_1),\ \bar{O}_1 = \frac{1}{(\xi')^2}(O_1+\tfrac{1}{2}\eta C_1).$$

Therefore four relative covariants in their canonical form are

$$P_1,\ C_1,\ E_1+4O_1,\ 2J_1E_1-C_1J'_1.$$

By converting these expressions into the original coefficients and variables we find the complete system of covariants for (A) to be

$$P,\ C,\ C_3 = E+4(O-\tfrac{1}{2}IP) = E+2N,\ C_7 = 2\theta_4E-\theta'_4C.$$